数理解析研究所講究録
1106 巻 1999 年 213-224

An MR-complete system S and its functional interpretation

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1 Introduction

In a previous work [5], Yasugi and Hayashi formulated a system of *constructive arithmetic* with *transfinite recursion* and *bar induction.* This system was called **TRDB**, which is a streamlined version of the system used by Yasugi in [4] to prove the accessibility of an order system.

TRDB is, however, a mathematically interesting system on its own right. For this reason, it has been studied from various aspects (see [4], [5], [6], [1] and [2]). The present article is a sequel to these preceding works. In this paper, we deal with the *modified realizability* (abbreviated to *MR*) interpretation of a constructive arithmetic corresponding to the interpretation of **TRDB** in **TRM** obtained in [5].

A system **S** is said to be complete with respect to an interpretation R if **S** interprets itself with respect to R. As for the first order constructive arithmetic, its extensions which are complete with regards to some interpretations are already known (see, for example, Theorem 3.4.8. of [3]). We questioned if there be an extension of **TRDB** which is complete with respect to *MR*-interpretation, and have reached a conclusion.

It is also mentioned that **TRM** interprets **S**, that is, an algorithm inherent in a proof of **S** is realized as a functional in **TRM**. This implies that **S** is of the same algorithmic strength as **TRDB**.

In this paper, we omit proofs of all theorems. The details will be published elsewhere.

2 Preliminaries

In order to facilitate the reader to understand this article, we first give a concise presentation of the system **TRDB**, as well as of the notion called *type-form.* The definitions are quoted mostly from the article [5] written by Yasugi and Hayashi.

Our theory depends on a pre-supplied, primitive recursive well-ordered structure $\mathcal{I} \equiv (I, <_I)$ on natural numbers. For the sake of simplicity, we assume the order type of I is less than ε_0.

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Definition 2.1 (**Basic language $\mathcal{BL}$**)
(1) The language $\mathcal{BL}$ consists of the following.
(1. 1) Propositional connectives $\wedge$ (*and*), $\supset$ (*imply*).
(1. 2) n-ary variables $x_0^n, x_1^n, x_2^n, \cdots, x_p^n, \cdots$. For $n = 0$, x_p^n is a variable which ranges over natural numbers. It will be called a number variable.
(1. 3) Constant symbols for all the number-theoretic functions which are primitive recursive in function parameters.
(1. 4) Predicate constant symbol $=$.
(2) $\mathcal{BL}$-terms are defined by (2.1) $\sim$ (2.4) below.
(2. 1) A constant or a variable of $\mathcal{BL}$ is a term of its arity.
(2. 2) If f is an n-ary term, and if $t_1, \cdots, t_n$ are number terms, then $f(t_1 \cdots t_n)$ is a number term.
(3) Every atomic formula of $\mathcal{BL}$-language is of the form $s = t$, where s and t are number terms. The $\mathcal{BL}$-formulas are defined from atomic formulas by applications of the propositional connectives.

Let c be a new, unary function constant symbol. We can extend $\mathcal{BL}$ to $\mathcal{BL}(c)$ adding c to it.

Using $\mathcal{BL}(c)$-language, one can define the system **TRDB**, as in Takaki [1].

TRDB may be considered to be **HA** (Heyting arithmetic) with definition by *transfinite recursion* and *bar induction*.

Definition 2.2 (**System TRDB**)
Symbols and Terms
(1) All symbols and terms of $\mathcal{BL}(c)$-language serve as those of **TRDB**.
(2) Special predicate constants H and Σ_H.
(3) Logical symbols $\wedge$, $\supset$, $\forall$ and $\exists$.
Formulas
(1) $s = t$ is an atomic formula of **TRDB**, where s and t are number terms.
(2) $H(s,\ t)$ and $\Sigma_H(s,\ s',\ t)$ are atomic formulas of **TRDB**, where s, s' and t are number terms.
(3) If A and B are formulas, then $A \wedge B$, $A \supset B$ and $\forall x A$ are formulas, where x is a variable.
(4) If A is a formula, then $\exists x A$ is a formula, where x is a number variable and x does not occur in any subformula of A which is of the form either $H(i,\ t)$ or $\Sigma_H(j,\ i,\ t)$. This restriction on the application of $\exists$ is called the admissibility.
Axioms and inference rules
(1) **TRDB** contains inference rules of constructive logic formulated in natural deductions as usual.
(2) **TRDB** contains axioms and inference rules on constants of $\mathbf{PRA}^2$ (primitive recursive arithmetic with function variables).
(3) **TRDB** contains *monotone bar induction* as explained below. Let $R[a]$ be a formula with a number variable a free, and suppose $R[a]$ contains neither any

quantifier, H, Σ_H, nor any variable except a. Such a formula will be called *elementary.*

Let $R[a]$ be an elementary formula. Then $R[a]$ is said to be ***monotone*** if $R[a]$ satisfies $\forall f \exists n\ R[f\lceil n]$ and $\forall f\ \forall m < n\ (R[f\lceil m] \Rightarrow R[f\lceil n])$. Now, our ***bar induction*** can be expressed as follows:

$$\frac{\begin{matrix}\vdots \\ \forall z(R[z] \supset A[z])\end{matrix} \qquad \begin{matrix}\vdots \\ \forall z(\forall x A[z * x] \supset A[z])\end{matrix}}{A[t]}\,BI,$$

where $A[a]$ is an arbitrary formula, $R[a]$ is an arbitrary monotone formula and t is an arbitrary number term.
(4) **TRDB** contains ***definition by transfinite recursion*** $TRD(G, \mathcal{I})$. We fix a formula $G[a,\ b]$, where a and b are free number variables and G satisfies the following conditions.
(i) No free variable occurs of G except a or b.
(ii) Predicate constant H does not occur in G, and every Σ_H in G occurs in the form $\Sigma_H(j,\ a,\ s)$, where j and s are some terms.

The axiom $TRD(G, \mathcal{I})$ stands as follows:

$$\forall x \forall y\ (H(x,\ y) \Leftrightarrow G(x,\ y)).$$

Type-forms are types with parametric variables. They are briefly described below. See Section 2 and Section 4 of [5] for detail.

Definition 2.3 (Type-form)
Type-forms are defined below, based on the language $\mathcal{BL}$.
(1) Symbols N and $\mathbf{1}$ are (atomic) type-forms.
(2) If α and β are type-forms, then so are $\alpha \to \beta$ (*function space*), $\{x\}\alpha$ (*parametric abstraction*), $\pi(\alpha;\ t)$ (*projection*) and $cond[A;\ \alpha,\ \beta]$ (*case definition*), where x is a variable of $\mathcal{BL}$, t is a $\mathcal{BL}$-term and A is a $\mathcal{BL}$-formula.
(3) $\mathcal{R}[i,\ t]$ (*transfinite recursion*) is a type-form, where $\mathcal{R}$ is a special letter denoting recursion operator, and i and t are number $\mathcal{BL}$-terms. ($\mathcal{R}[i,\ t]$ is characterized by a fixed type-form η. See (1. 4) of Definition 2.4 below.)
(4) $\rho[j <_I i;\ \mathcal{R}[j,\ s]]$ (*restriction*) is a type-form, where i, j and s are number $\mathcal{BL}$-terms.

Definition 2.4 (Conversion of type-form)
(1) Conversion rules of type-forms, say α to β, denoted by $\alpha \Rightarrow \beta$, are the following.
(1. 1) $\pi(\{x\}\alpha; t) \Rightarrow \alpha[t/x]$, where $\alpha[t/x]$ represents the substitution of t for x in α.

(1. 2) $cond[A;\alpha_1,\alpha_2] \Rightarrow \beta$, where β is α_1 if A is a true sentence in the standard interpretation of the symbols, and β is α_2 if A is a false sentence.
(1. 3) $\rho[j <_I i; \mathcal{R}[j,s]] \Rightarrow \beta$, where β is $\mathcal{R}[j,s]$ if $j <_I i$ is a true sentence, and it is $\mathbf{1}$ if $j <_I i$ is a false sentence.
(1. 4) Let Ξ be a designated symbol which is temporarily regarded as an atomic type. Let η be a type-form with Ξ but without $\mathcal{R}$, whose free variables are k and w, where k and w are of arity 0 and Ξ contains only k as its free variable. $\eta[i/k,t/w]$ will denote simultaneous substitution of i for k and t for w, and $\mathcal{R}(i)$ will abbreviate $\{j\}\{x\}\rho[j <_I i; \mathcal{R}[j,x]]$.

$$\mathcal{R}[i,t] \Rightarrow \eta[i/k,t/w][\mathcal{R}(i)/\Xi],$$

where $X[Y/\Xi]$ represents the substitution of Y for Ξ in X.
(2) α is said to be 1-reduced to β if β is obtained from α by one conversion applied to a subtype of α (called a reduction of α to β), with the strategy that the conversions (1. 3) enjoys a following property: one does not go inside $\rho[j <_I i;\ \mathcal{R}[j,\ t]]$. We call this ***ρ-strategy***.
(3) A type-form is said to be normal if no conversion rule applies to it.

Remark 2.5 The conversion rule of $\mathcal{R}[i,\ t]$ is determined by a type-form η in Definition 2.4 (1. 4). We call such a type-form η a *central* type-form.

Theorem 2.6 **(Strong normalizability of type-forms:** Yasugi and Hayashi, Theorem 1 in [5])
Every type-form is strongly normalizable to a unique normal form (under the ρ-strategy), that is, any process of reductions ends up with a normal type-form. Furthermore, there is a unique such normal type-form for every type-form.

In what follows, we let $\alpha \sim \beta$ mean that α and β have the same normal form. We abbreviate $app(\alpha;\ t)$ to αt, $\{l\}cond[l = 0;\ \alpha,\ cond[l = 1;\ \beta,\ 1]]$ to $\alpha \times \beta$, and $\pi(\gamma;\ i)$ to $\pi_{i+1}\gamma$ for $i = 0$ or 1. $\alpha \times \beta$ represents the product types.

We define a mapping $[\![\ ]\!]$ of admissible formulas (of **TRDB**) into type-forms as well as specify the central type-form. This mapping is introduced by Yasugi and Hayashi in Definition 8.1 of [5].

Definition 2.7 **(Interpretation $[\![\ ]\!]$)**
To each (admissible) formula A (cf. Definition 2.2), we associate a type-form $[\![A]\!]$, as follows.
(1) If A is free of $\exists$ and H, then $[\![A]\!] = \mathbf{1}$.
(2) Suppose A is free of H but contains $\exists$.
(2. 1) $[\![\exists xB]\!] = N$ if $[\![B]\!] = \mathbf{1}$; $[\![\exists xB]\!] = N \times [\![B]\!]$ otherwise.
(2. 2) $[\![\forall xB]\!] = \mathbf{1}$ if $[\![B]\!] = \mathbf{1}$; $[\![\forall xB]\!] = \{x\}[\![B]\!]$ otherwise.
(2. 3) $[\![B \supset C]\!] = [\![C]\!]$ if $[\![B]\!] = \mathbf{1}$ or $[\![C]\!] = \mathbf{1}$; $[\![B \supset C]\!] = [\![B]\!] \to [\![C]\!]$ otherwise.
(2. 4) $[\![B \wedge C]\!] = [\![C]\!]$ if $[\![B]\!] = \mathbf{1}$; $[\![B \wedge C]\!] = [\![B]\!]$ if $[\![C]\!] = \mathbf{1}$; $[\![B \wedge C]\!] = [\![B]\!] \times [\![C]\!]$ otherwise.

(3) Let A be any admissible formula. We define $[\![A]\!]$ as follows.
(3. 1) $[\![H(i,\ a)]\!] = \mathcal{R}[i,\ a]$. $[\![\Sigma_H(j,\ i,\ b)]\!] = \rho[j <_I i;\ \mathcal{R}[j,\ b]]$.
(3. 2) For $\forall$, $\supset$ and $\wedge$, follow (2) above.
(4) When reductions of type-forms are concerned, we need to specify the central type-form η in Definition 2.4 (1. 4). Let G be the formula used in $TRD(G,\mathcal{I})$, and let G' be a formula-like expression obtained from G by replacing H and Σ_H with new symbols Ξ and Σ_Ξ respectively. Define $[\![\Xi(i,\ a)]\!] = \Xi(i,\ a)$ and $[\![\Sigma_\Xi(j,\ i,\ a)]\!] = \rho[j <_I i;\ \Xi(j,\ b)]$. Now apply (2) above to G', and put $\eta = [\![G']\!]$.

3 An extended system S

In this section, we introduce a system which is an extension of **TRDB**. This system, called **S**, was first considered in [2] so that **S** be complete with respect to an extended modified realizability interpretation.

Preceding the definition of **S**, we define a notation of certain type-forms.

Definition 3.1 (**Number-theoretic type-form** $N(n)$)
For a natural number n, we define an expression $N(n)$ of a type-form, which is defined by: (i) $N(0) = N$; (ii) $N(n+1) = N \to N(n)$.

The system **S** is defined by the following Definitions 3.2 and 3.5.

Definition 3.2 (**Language of** $\mathbf{S}(G,\mathcal{I})$)
Given a system $\mathbf{TRDB}(G,\mathcal{I})$, we define a language of an extended system $\mathbf{S}(G,\mathcal{I})$ (often abbreviated to **S**) as follows.
(1) **Symbols**
(1. 1) All symbols of $\mathcal{BL}(c)$-language are those of **S**.
(1. 2) For each type-form α, we prepare variables (called variable-forms) $X_1^\alpha, X_2^\alpha, \ldots$.
(1. 3) $*$, *app*, *cond*, ( ) and [ ] are operational symbols, which are used in order to construct terms of **S**.
(1. 4) $\mathcal{H}$ and $\Sigma_\mathcal{H}$ are special predicate constants of **S**.
(2) **Terms**
We induce terms of **S** in the following (2. 1)~(2. 8). We fix a central type-form η as in Definition 2.7 (4). (See also Definition 2.4 and Remark 2.5.)

By $\phi : \alpha$, we express the fact that ϕ is a term of a type-form α.
(2. 1) $* : \mathbf{1}$.
(2. 2) For any variable-form X_n^α, $X_n^\alpha : \alpha$.
Note. We distinguish "$\mathcal{BL}(c)$-variables" and "variable-forms." In **S**, Variable-forms rule on usual variables, and $\mathcal{BL}(c)$-variables rule on parameters in **S**-terms.
(2. 3) For every natural number n, $id_n : \{x^n\}N(n)$, where x^n is an n-ary $\mathcal{BL}(c)$-variable. We use this term in order to construct **S**-terms corresponding to $\mathcal{BL}(c)$-terms. See also Remark 3.3 (1) below.

(2. 4) $\phi : \beta$, whenever $\phi : \alpha$ and β is a type-form with $\alpha \sim \beta$ (α and β have the same normal form).
(2. 5) $app(\phi;\ t) : \alpha[t/x]$, whenever $\phi : \{x\}\alpha$ and t is a $\mathcal{BL}(c)$-term whose arity is the same as that of the $\mathcal{BL}(c)$-variable x.
(2. 6) $app(\phi;\ \psi) : \beta$, whenever $\phi : \alpha \to \beta$ and $\psi : \alpha$.
(2. 7) $cond[A;\ \phi,\ \psi] : cond[A;\ \alpha,\ \beta]$, if $\phi : \alpha$, $\psi : \beta$ and if A is a $\mathcal{BL}(c)$-formula.
(2. 8) $\lambda x.\phi : \{x\}\alpha$ if $\phi : \alpha$.
We call ϕ a *number* term if $\phi : N$.
(3) **Formulas**
(3. 1) $\phi = \psi$ is an atomic formula of **S**, where ϕ and ψ are number terms.
(3. 2) $\mathcal{H}(i,\ t,\ \phi)$ is a formula of **S**, if i and t are number $\mathcal{BL}(c)$-terms and if ϕ is an **S**-term with the type-form α satisfying $\alpha \sim \mathcal{R}[i,\ t]$.
(3. 3) $\Sigma_{\mathcal{H}}(j,\ i,\ t,\ \phi)$ is a formula of **S**, where i, j and t are number $\mathcal{BL}(c)$-terms, and ϕ is an **S**-term with the type-form α satisfying $\alpha \sim \rho[j <_I i;\ \mathcal{R}[i,\ t]]$.
(3. 4) $A \wedge B$ and $A \supset B$ are formulas if A and B are formulas of **S**.
(3. 5) Let A be a formula of **S** and x is a $\mathcal{BL}(c)$-variable. Then, $\forall x A$ is a formula of **S**.
(3. 6) Let A be a formula of **S**, and let X^α be a variable-form. Then, $\forall X^\alpha A$ and $\exists X^\alpha A$ are formulas of **S** if X^α in A satisfies the following condition $\Gamma(X^\alpha, A)$:

$\Gamma(X^\alpha, A)$: If there is a free occurrence of X^α in A, then all $\mathcal{BL}(c)$-variables occurring in α freely are free in A. (That is, if x is a free $\mathcal{BL}(c)$-variable occurring in α, then such an occurrence of X^α is not in the scope of any $\forall x$ in A.)

Note that $\exists x$ is *not* admitted in **S**-formulas. This is because, we do not need to bound $\mathcal{BL}$-variables by $\exists$-quantifier due to the admissibility of **S**- (**TRDB**-) formulas (cf. Definition 2.2 and Remark 4.2 of [2]).

Remark 3.3 For an n-ary $\mathcal{BL}(c)$-term t, the intended meaning of the **S**-term $app(id_n;\ t)$ is t. Note that $app(id_n;\ t)$ has the type-form $N(n)$ defined in Definition 3.1. In what follows, we abbreviate $app(id_n;\ t)$ to $\mathbf{t}$.

Definition 3.4 (**Morphism** $*$)
We define a morphism $*$ which sends a formula of **TRDB** to a certain formula of **S** in the following (1) and (2).
(1) Let A be a formula of **TRDB**, and let t be a $\mathcal{BL}(c)$-term in A. We define an **S**-term $S(t;\ A)$ by induction on the construction of t as follows.
(1. 1) Let t be an n-ary $\mathcal{BL}(c)$-variable x_i^n.
(i) If x_i^n is bound by an $\exists$-quantifier in A, then $S(x_i^n;\ A) = X_i^N$. Note that, in this case, the arity n of x_i^n is 0 by the admissibility of **TRDB**-formula.
(ii) Otherwise, $S(x_i^n;\ A) = app(id_n;\ x_i^n)$.
(1. 2) If t is an n-ary function constant f, then $S(f;\ A) = app(id_n;\ f)$.

(1. 3) If t is a number $\mathcal{BL}(c)$-term $f(t_1, \cdots, t_n)$ defined in Definition 2.1 (2. 2), then $S(f(t_1, \cdots, t_n);\ A) = app(app(\cdots app(S(f;\ A);\ S(t_1;\ A)); \cdots);\ S(t_n;\ A))$.
(2) Let A be a formula of **TRDB**, and let B be a subformula of A. We define an **S**-formula $S(B;\ A)$ by induction on the construction of B, as follows.
(2. 1) If B is an atomic formula which is not an H-formula, then $S(B;\ A)$ is the formula obtained from B by replacing all terms $t_1, \cdots, t_n$ of B by the terms $S(t_1;\ A), \cdots, S(t_n;\ A)$.
(2. 2) $S(H(i,\ a);\ A) = \exists X^{\mathcal{R}[i,\ a]}\mathcal{H}(i,\ a,\ X^{\mathcal{R}[i,\ a]})$.
(2. 3) $S(\Sigma_H(j,\ i,\ a);\ A) = \exists X^{\rho[j<_I i;\ \mathcal{R}[j,\ a]]}\Sigma_{\mathcal{H}}(j,\ i,\ a,\ X^{\rho[j<_I i;\ \mathcal{R}[j,\ a]]})$.
(2. 4) $S(B \wedge C;\ A) = S(B;\ A) \wedge S(C;\ A)$.
(2. 5) $S(B \supset C;\ A) = S(B;\ A) \supset S(B;\ A)$.
(2. 6) $S(\forall x B[x];\ A) = \forall x S(B[x];\ A)$.
(2. 7) $S(\exists x B[x];\ A) = \exists X^N S(B[x];\ A)$, where $X^N = S(x;\ A)$.
Now, for every **TRDB**-formula A, we define A^* by: $A^* = S(A;\ A)$.

In what follows, we abbreviate an **S**-term $\lambda x.cond[x = 0;\ \phi,\ cond[x = 1;\ \psi,\ *]]$ (*pairing*) to $\langle \phi,\ \psi \rangle$, and $app(\phi, i)$ (*projection*) to $\pi_{i+1}(\phi)$ for $i = 0, 1$.

Definition 3.5 **(Axioms and inference rules of S)**
(1) Axioms and inference rule of **S** as constructive logic are defined similarly to those of 2.2.
(2) Axioms and inference rules with respect to elementary arithmetic are obtained from those of **TRDB** by $*$-mapping.
(3) Bar induction of **S** is the same as that of **TRDB**.
(4) $TRD^*(G, \mathcal{I})$:

$$\forall x \forall y\ (\ \exists X^{\mathcal{R}[x,\ y]}\mathcal{H}(x,\ y,\ X^{\mathcal{R}[x,\ y]}) \Leftrightarrow G^*[x,\ y]\ ).$$

Here, $G[a,\ b]$ is the formula defined in $TRD(G, \mathcal{I})$ of **TRDB** (cf. Definition 2.2), and $G^*[a,\ b]$ is the formula obtained from $G[a,\ b]$ by the morphism $*$.
(5) Implication axiom:

$$(A \supset \exists Y^\beta B[Y^\beta]) \Rightarrow \exists Y^\beta (A \supset B[Y^\beta]),$$

where A is $\exists$-free and B is any formula.
(6) Axiom of choice:

$$\forall \mathbf{X} \exists Y^\beta A[\mathbf{X},\ Y^\beta] \Rightarrow \exists Z^\gamma \forall \mathbf{X} A[\mathbf{X},\ Z^\gamma \mathbf{X}],$$

where A is an $\exists$-free formula, $\mathbf{X}$ is either a $\mathcal{BL}(c)$-variable x (in which case γ is $\{x\}\beta$) or a variable-form X^α (in which case γ is $\alpha \to \beta$).
(7) Product axiom:

$$\exists X^\alpha \exists Y^\beta A[X^\alpha,\ Y^\beta] \Rightarrow \exists Z^{\alpha\times\beta} A[\pi_1(Z^{\alpha\times\beta})/X^\alpha,\ \pi_2(Z^{\alpha\times\beta})/Y^\beta],$$

where A is $\exists$-free.

Remark 3.6 (1) Note that we restrict the axioms for equality of term-forms with type-form N. For example, **S** contains the axiom of the form $\mathbf{t} = \mathbf{t}$, but does not contain $X^N = X^N$.
(2) In what follows, the bold-script upper case alphabet $\mathbf{X}$ expresses a variable-form X^α with the type-form α or a $\mathcal{BL}(c)$-variable x. Similarly with $\mathbf{Y}$ and $\mathbf{Z}$.

Proposition 3.7 (Embedding of TRDB into S)
TRDB $\vdash A$ implies $\mathbf{S} \vdash A^*$ for any formula A. (See Definition 3.4 for $*$.)

Next we define an interpretation MR on formulas of **S**.

Definition 3.8 (Modified realizability interpretation MR)
We define the modified realizability interpretation MR, which translates **S**-formulas to certain **S**-formulas. Given a formula A, we define $MR(A)$ by induction on the construction of A. $MR(A)$ will be of the form $\exists W^\delta \mathcal{A}[W^\delta]$, where $\mathcal{A}[W^\delta]$ is an $\exists$-free formula. (In the case (1. 1) below, $\exists X$ is vacuous.)
(1) Suppose that A does not contain $\mathcal{H}$.
(1. 1) If A does not contain $\exists$-quantifier, then $MR(A) = A$.

In (1. 2)~(1. 5) below, A is assumed to contain an $\exists$-quantifier. We assume the induction hypotheses $MR(B) = \exists Y^\beta \mathcal{B}[Y^\beta]$ and $MR(C) = \exists Z^\gamma \mathcal{C}[Z^\gamma]$, where $\mathcal{B}$ and $\mathcal{C}$ do not contain $\exists$-quantifiers.
(1. 2) If $A = \exists X^\alpha B$, then $MR(\exists X^\alpha B) = \exists W^{\alpha\times\beta}\mathcal{B}[\pi_1 W^{\alpha\times\beta}/X^\alpha, \pi_2 W^{\alpha\times\beta}/Y^\beta]$. If $MR(B)$ is an $\exists$-free formula $\mathcal{B}$, then $MR(A) = \exists X^\alpha \mathcal{B}$.
(1. 3) If $A = B \wedge C$, then $MR(B\wedge C) = \exists W^{\beta\times\gamma}(\mathcal{B}[\pi_1 W^{\beta\times\gamma}/Y^\beta] \wedge \mathcal{C}[\pi_2 W^{\beta\times\gamma}/Z^\gamma])$. In the case where $MR(B)$ is $\exists$-free, $MR(A) = \exists Z^\gamma(\mathcal{B} \wedge \mathcal{C}[Z^\gamma])$. If $MR(C)$ is $\exists$-free, then $MR(A) = \exists Y^\beta(\mathcal{B}[Y^\beta] \wedge \mathcal{C})$.
(1. 4) If $A = B \supset C$, then $MR(B \supset C) = \exists W^{\beta\to\gamma}\forall Y^\beta(\mathcal{B}[Y^\beta] \supset \mathcal{C}[W^{\beta\to\gamma}Y^\beta/Z^\gamma])$. If $MR(B)$ is $\exists$-free, then $MR(A) = \exists Z^\gamma(\mathcal{B} \supset \mathcal{C}[Z^\gamma])$. If $MR(C)$ is $\exists$-free, then $MR(A) = \forall Y^\beta(\mathcal{B}[Y^\beta] \supset \mathcal{C})$.
(1. 5) If $A = \forall \mathbf{X} B$, then $MR(\forall \mathbf{X} B) = \exists W^\delta \forall \mathbf{X}\mathcal{B}[\mathbf{X},\ W^\delta \mathbf{X}/Y^\beta]$. Here δ is determined as follows: δ is $\{x\}\beta$ if $\mathbf{X}$ is a $\mathcal{BL}(c)$-variable x; δ is $\alpha \to \beta$ if $\mathbf{X} = X^\alpha$. If $MR(B)$ is $\exists$-free, then $MR(A) = \forall \mathbf{X}\mathcal{B}$.
(2) A contains $\mathcal{H}$ as its subformula, that is, A is an $\mathcal{H}$-formula.
(2. 1) We define the MR-interpretation of basic $\mathcal{H}$-formulas as follows:

$$MR(\mathcal{H}(i,\ t,\ X^\alpha)) = \mathcal{H}(i,\ t,\ X^\alpha).$$

$$MR(\Sigma(j,\ i,\ t,\ X^\alpha,\ \mathcal{H})) = \Sigma(j,\ i,\ t,\ X^\alpha,\ \mathcal{H}).$$

(2. 2) For the general cases of formulas which contain $\mathcal{H}$, the MR-interpretations can be defined from (2. 1) by applying (1. 1)~(1. 5).

In [2], we investigated the following theorem.

Theorem 3.9 (Completeness theorem for MR-interpretation.)
Let A be a formula in **S**. Then

$$\mathbf{S} \vdash MR(A) \Leftrightarrow A.$$

4 Term-forms: review

In this section, we present the definition of *term-forms* (terms with parameter types) and reduction rules of them, which were introduced by Yasugi and Hayashi in [5] and [6]. We repeat the definitions in some detail for the reader's convenience.

Definition 4.1 (**Term-forms**: See also Definition 4.1 of [5].)
Term-forms are defined below. $\phi : \alpha$ will express that term-form ϕ is of type-form α. If $\alpha \sim \beta$ (cf Theorem 2.6), then we let $\phi : \alpha$ imply $\phi : \beta$.
(1) $*$ is an atomic term-form whose type-form is $\mathbf{1}$.
(2) For a natural number n, id_n is an atomic term-form whose type-form $\{x^n\}\tau(n)$. Here x^n is an n-ary $\mathcal{BL}(c)$-variable.
(3) For each natural number n and for each type-form β, X_n^β (the nth variable-form of type-form β) is an atomic term-form.
(4) If $\phi : \gamma$, then $\lambda x.\phi : \{x\}\gamma$. If $\phi : \gamma$, then $\lambda X_n^\beta.\phi : \beta \to \gamma$, where the condition $\Gamma^*(\phi,\ \beta)$ below is assumed.

$\Gamma^*(\phi,\ \beta)$: For each X_n^δ, where $\delta \sim \beta$ and X_n^δ occurs freely in ϕ, no free $\mathcal{BL}(c)$-variable x in δ is bound by λx in ϕ.

(5) If $\phi : \{x\}\gamma$, then $app(\phi;\ t) : \gamma[t/x]$. If $\phi : \beta \to \gamma$ and $\psi : \beta$, then $app(\phi;\ \psi) : \gamma$. We abbreviate these as ϕt and $\phi\psi$ respectively.
(6) If $\phi : \beta$ and $\psi : \gamma$, then $cond[A;\ \phi,\ \psi] : cond[A;\ \beta,\ \gamma]$.
(7) If $\phi : \mathcal{R}[j,\ s]$, then $\sigma[j <_I i;\ \phi] : \rho[j <_I i;\ \mathcal{R}[j,\ s]]$.
(8) A functional constant μ of type $\{f\}N$, where f is a unary function $\mathcal{BL}(c)$-variable. (μ represents a modulus of finiteness functional of the order $<_I$. See the beginning of Preliminaries.)
(9) Bar recursion. Let $\mathbf{b}$ stand for a continuous (bar recursive) functional for a neighborhood function (see [5]). If $\phi : \{z\}\gamma$ and $\psi : \{z\}(\{s\}\gamma[z * s/z] \to \gamma)$, then $\mathcal{B}[\mathbf{b};\ \phi,\ \psi;\ m,\ \ f] : \gamma[f\lceil m/z]$, where m and f are $\mathcal{BL}$-terms of arities respectively 0 and 1, and $f\lceil m$ represents the restriction of f to the domain m.

Notice that the terms of $\mathbf{S}$ defined in Definition 3.2 form a subset of these term-forms. The language of $\mathbf{S}$ is *not* extended to include these term-forms.

Definition 4.2 (**Conversion of term-forms**: See Definition 4.3 of [5].)
(1) Conversion rules of term-forms are the following.
(1. 1) $(\lambda x.\phi)t \Rightarrow \phi[t/x]$.
(1. 2) $(\lambda X_n^\beta.\phi)\psi \Rightarrow \phi[\psi/X_n^\delta]$, where $\beta \sim \delta$ and X_n^δ is free in ϕ. (ψ is substituted for X_n^δ for any such X_n^δ.)
(1. 3) $cond[A; \phi, \psi] \Rightarrow \chi$, where χ is ϕ if A is a true sentence and χ is ψ if A is a false sentence.
(1. 4) $\sigma[j <_I i; \phi] \Rightarrow \chi$, where χ is ϕ if $j <_I i$ is a true sentence and χ is $*$ if $j <_I i$ is a false sentence.

(1. 5) The conversion rule for $\mathcal{B}$ is as follows.

$$\mathcal{B}[\mathbf{b};\phi,\psi,m,f] \Rightarrow app(\phi;\ f\lceil m),$$

if $\mathbf{b}(f) \leq m$ is a true sentence with $\mathbf{b}$ added to $\mathcal{BL}$, and

$$\mathcal{B}[\mathbf{b};\phi,\psi,m,f] \Rightarrow app(app(\psi;\ f\lceil m);\ \lambda k.\mathcal{B}[\mathbf{b};\ \phi,\psi,m+1,(f\lceil m)\#k])$$

if $\mathbf{b}(f) > m$ is a true sentence. Here $(f\lceil m)\#k$ is a unary term being an extension of the finite sequence $f\lceil m$ of the form $\langle f(0),\ f(1),\cdots,\ f(m-1),\ k,\ 0,\ 0,\cdots\rangle$.
(1. 6) Let s and t be $\mathcal{BL}(c)$-terms. If t is the result of computation of s, then $t \Rightarrow s$.
(2) We can define "ϕ 1-reduces to ψ" and "ϕ is reducible to ψ" similarly to Definition 2.4, with the strategies that the conversions of σ and $\mathcal{B}$ have priorities of reduction. A term-form is said to be *normal* if it can not be reduced any longer.

Definition 4.3 **(Term system TRM)**
TRM is the system consisting of type-forms and term-forms together with their reductions.

Theorem 4.4 **(Strong normalizability of TRM**; see Theorem 3 in [5])
Every term-form is strongly normalizable to a unique normal form (under the σ- and $\mathcal{B}$-strategies).

5 Semantics for S-formulas

In this section, we define a truth-value ET of **S**-formulas under certain term-forms. As our main objective, we present the main theorem of this paper, which assures validity of **S** with respect to the truth-value ET.

Definition 5.1 **(Degree of S-formula)**
(1) For a primitive recursive order structure $\mathcal{I} = (I,\ <_I)$, which we assumed in defining **TRDB**, we define $\mathcal{I}^* = (I^*,\ <^*)$ as in [5] and [1].

$$I^{\sim} = \{i^{\sim}\ ;\ i \in I\}\ ;\ I^* = I \cup I^{\sim} \cup \{\infty\}\ ;\ i <^* i^{\sim} <^* j <^* \infty \text{ when } i <_I j.$$

Moreover, we put $\mathcal{I}_* = \omega^{\mathcal{I}^*}$, where we identify $\mathcal{I}^*$ with its order type.
(2) Let A be either $\mathcal{H}(i,\ t,\ X^\alpha)$ or $\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^\alpha)$. We define the *rank* $r(A)$ of A as follows.
(i) $r(\mathcal{H}(i,\ t,\ X^\alpha)) = i^{\sim}$ if i is closed; $r(\mathcal{H}(i,\ t,\ X^\alpha)) = \infty$ otherwise.
(ii) $r(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^\alpha)) = j$ if j is closed; $\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^\alpha)) = \infty$ otherwise.
(3) Let A be a formula. Then, we define the *degree* $d(A)$ of A as follows.
(i) If A is an atomic formula except an $\mathcal{H}$-formula, then $d(A) = 1$.
(ii) $d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max(d(B),\ d(C)) + 1$.

(iii) $d(\forall \mathbf{X} B[\mathbf{X}]) = d(\exists \mathbf{X} B[\mathbf{X}]) = d(B[\mathbf{X}]) + 1$, where $\mathbf{X}$ is a $\mathcal{BL}(c)$-variable or a variable-form.
(iv) $d(\mathcal{H}(i,\ t,\ X^{\alpha})) = \omega^{r(\mathcal{H}(i,\ t,\ X^{\alpha}))}$; $d(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha})) = \omega^{r(\Sigma_{\mathcal{H}}(i,\ j,\ t,\ X^{\alpha}))}$.

Definition 5.2 (**Semantics**)
Let $A[X_1^{\alpha_1}, \cdots,\ X_n^{\alpha_n},\ x_1, \cdots,\ x_m]$ be an $\exists$-free formula of **S**, where $X_1^{\alpha_1}, \cdots,$ $X_n^{\alpha_n}$ are all free variable-forms of A and $x_1, \cdots,\ x_m$ are all free $\mathcal{BL}(c)$-variables of A. We abbreviate $A[X_1^{\alpha_1}, \cdots,\ X_n^{\alpha_n},\ x_1, \cdots,\ x_m]$ to $A[\vec{X}, \vec{x}]$.

Let $s_1, \cdots,\ s_m$ be closed $\mathcal{BL}(c)$-terms, where each s_k has the same arity as that of x_k, and let $\phi_1, \cdots, \phi_n$ be only-$\vec{x}$-open term-forms, where each ϕ_j has a type-form β_j with $\alpha_j[\vec{s}/\vec{x}] \sim \beta_j[\vec{s}/\vec{x}]$. We define the truth-value of $A[\vec{X}, \vec{x}]$ under environments of $\phi_1, \cdots, \phi_n$ for $X_1^{\alpha_1},\ \cdots,\ X_n^{\alpha_n}$ and $s_1, \cdots, s_m$ for $x_1,\ \cdots,\ x_m$, which is denoted by $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s})$, by transfinite induction on the degree of $A[\vec{X}, \vec{x}][\vec{s}/\vec{x}]$.
(1) $A[\vec{X}, \vec{x}]$ does not contain H.
(1. 1) $A[\vec{X}, \vec{x}]$ is an atomic formula, that is, $A[\vec{X}, \vec{x}]$ is of the form $\Phi[\vec{X}, \vec{x}] = \Psi[\vec{X}, \vec{x}]$. We define the truth-value by: $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = T$ (*true*) if $\Phi(\vec{\phi}[\vec{s}/\vec{x}],$ $\vec{s})$ and $\Psi(\vec{\phi}[\vec{s}/\vec{x}], \vec{s})$ have the same normal form; $ET(A[\vec{X}, \vec{x}];\ \vec{\phi}, \vec{s}) = F$ (*false*) otherwise.
(1. 2) The connectives $\wedge$, $\supset$ and $\forall$ are interpreted classically.
(2) $A[\vec{X}, \vec{x}]$ contains $\mathcal{H}$.
(2. 1) $A[\vec{X}, \vec{x}]$ is of the form $\mathcal{H}(i,\ t,\ X_1^{\alpha_1})$. Let $\mathcal{G}[i,\ t,\ X^{\xi}]$ be the formula obtained from $MR(G^*[i,\ t]) = \exists X^{\xi} \mathcal{G}[i,\ t,\ X^{\xi}]$. We define the truth-value by

$$ET(\mathcal{H}(i,\ t\ X_1^{\alpha_1});\ \phi_1, \vec{s}) = ET(\mathcal{G}[i,\ t,\ X^{\xi}];\ \phi_1, \vec{s}).$$

(2. 2) $A[\vec{X}, \vec{x}]$ is of the form $\Sigma_{\mathcal{H}}(j,\ i,\ t,\ X_1^{\alpha_1})$.
(i) If $ET(j <_I i;\ \vec{s}) = T$, that is, $j[\vec{s}/\vec{x}] <_I i[\vec{s}/\vec{x}]$, then

$$ET(\Sigma_{\mathcal{H}}(j,\ i,\ t, X_1^{\alpha_1});\ \phi_1, \vec{s}) = ET(\mathcal{H}(\overline{j},\ \overline{t},\ X_1^{\mathcal{R}[\overline{j}, \overline{t}]});\ \phi_1[\vec{s}/\vec{x}]),$$

where $\overline{j} = j[\vec{s}/\vec{x}]$ and $\overline{t} = t[\vec{s}/\vec{x}]$.
(ii) If $ET(j <_I i;\ \vec{s}) = F$, then

$$ET(\Sigma_{\mathcal{H}}(j,\ i,\ t, X_1^{\alpha_1});\ \phi_1, \vec{s}) = F.$$

Note that $x <_I y$ is an $\mathcal{H}$-free formula with the intended meaning of the order of $\mathcal{I}$, which is a primitive recursive predicate. (See Definition 3.5 (2).)
(2. 3) If A is an $\mathcal{H}$-formula which is not atomic, then we follow the cases in (1).

The following theorem can be proved in a manner similar to the proof of the Main Theorem in Section 4 of [4], adding the new cases of implication axiom, axiom of choice, and product axiom.

Theorem 5.3 (Validity of S-theorem)
Let A $(= A[\vec{Y}, \vec{y}])$ be a theorem of **S**, where $\vec{Y}$ $(= Y_1^{\beta_1}, \cdots, Y_n^{\beta_n})$ are all free variable-forms of A and $\vec{y}$ $(= y_1, \cdots, y_m)$ are all $\mathcal{BL}(c)$-variables of A. If $MR(A[\vec{Y}, \vec{y}]) = \exists X^\alpha \mathcal{A}[X^\alpha, \vec{Y}, \vec{y}]$, then there exists a term-form Φ $(= \Phi[\vec{Y}, \vec{y}])$ satisfying the following:
(i) Φ has no free variable (-form) except $\vec{Y}, \vec{y}$. (That is, Φ is only-$\vec{y}$-open.)
(ii) For all closed $\mathcal{BL}(c)$-terms $\vec{t}$ $(= t_1, \cdots, t_m)$, where the arity of t_j is the same as that of y_j, and for all only-$\vec{y}$-open term-forms $\vec{\psi}$ $(= \psi_1, \cdots, \psi_n)$, where ψ_i has a type-form β'_i with $\beta'_i[\vec{t}/\vec{y}] \sim \beta_i[\vec{t}/\vec{y}]$, it holds that

$$ET(\mathcal{A}[X^\alpha, \vec{Y}, \vec{y}];\ \Phi[\vec{\psi}/\vec{Y}], \vec{\psi}, \vec{t}) = T.$$

As an immediate consequence of Theorem 5.3, we have the following result with respect to the existence of a function.

Corollary 5.4 For any Π_2^0-sentence of arithmetic, say $\forall x \exists y A[x,\ y]$, if $\forall x \exists y$ $A[x,\ y]$ is a theorem of **S**, then there exists a closed term-form Φ such $ET(\forall x A[x,$ $Yx];\ \Phi) = T$. That is, all functions which are provably total in **S** can be realized in **TRM**.

Corollary 5.5 **S** is consistent.

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